

A 1-LINKED LINK WHOSE LONGITUDES LIE IN THE SECOND COMMUTATOR SUBGROUP⁽¹⁾

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1. Introduction. In this paper we give an example of a link L of two polygonal simple closed curves in S^3 such that the longitudes of L lie in the second commutator subgroup, G'' , of its link group $G = \pi_1(S^3 - L)$, but L is 1-linked, that is the two simple closed curves of L do not bound disjoint orientable surfaces in S^3 . The question of the existence of such a link was raised by Eilenberg in [3] and again by Smythe in [5] and one motivation for this question is the observation that the longitudes of any boundary link lie in the second commutator subgroup of its link group. (A link is a boundary link if and only if it is not 1-linked, see Smythe [5]). In [2], [3], and [5] examples of 1-linking are given. In all these examples the authors proved 1-linking by showing that at least one of the longitudes was not in the second commutator subgroup. It is clear then that in our example L we must invent some other argument to show it is 1-linked.

2. The example L . In E^3 let T be the solid torus obtained by rotating the disk $x=0, (y-2)^2 + z^2 \leq 1$ about the z -axis. Let l_1 be the simple closed curve in S^3 consisting of the z -axis and the point at infinity. Figure 1 pictures the oriented

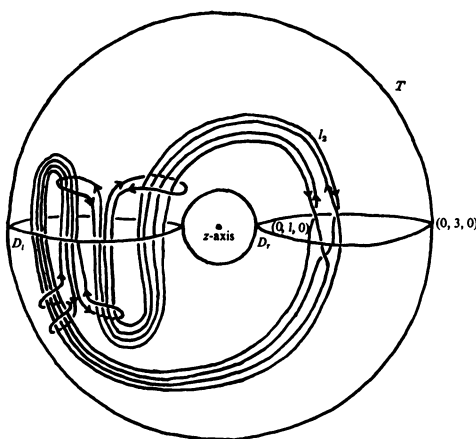


FIGURE 1

Presented to the Society, January 24, 1969; received by the editors January 20, 1969.

⁽¹⁾ Research for this paper was supported in part by the Old Gold Development Fund of the University of Iowa.

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simple closed curve l_2 in T as seen by looking down upon it from a point on the positive z -axis. (All the remaining figures in this paper will be drawn from the same viewpoint.) Our example is $L = l_1 \cup l_2$. Let $G = \pi_1(S^3 - L)$. To say the longitudes of L belong to G'' , the second commutator subgroup of G , we mean that on the boundary of each component of a regular neighborhood of L there is a simple closed curve (scc) lying in G'' (a loop in $S^3 - L$ actually determines a class of conjugate elements in G); for short we may say l_1 and l_2 belong to G'' or $L \in G''$. (See [4, p. 123] for a discussion of longitudes.)

Since l_2 is rather complicated some further sets will be needed. Let V be the neighborhood of l_2 in T illustrated by Figure 2; V is a cube with four handles

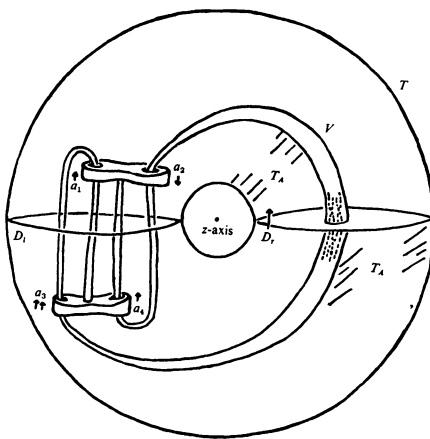


FIGURE 2

a_1, a_2, a_3 and a_4 . We may think of the a_i 's as disjoint annuli such that $\text{Bd}(\bigcup_{i=1}^4 a_i) \subset \text{Bd } V$ and $\text{Int}(\bigcup_{i=1}^4 a_i) \subset \text{Int } T - V$. Let a_1^+ denote the side of a_1 facing the same direction as the small arrow next to a_1 and a_1^- its opposite side. Define similarly a_2^+, a_3^+ , and a_4^+ , and their opposite sides a_2^-, a_3^- , and a_4^- . (See Figure 2.) Let $D_l = T \cap \{(x, y, z) : x=0, y < 0\}$ and $D_r = T \cap \{(x, y, z) : x=0, y > 0\}$, D_l and D_r are meridional disks of T . Choose the positive side of D_r , D_r^+ (negative side D_r^-) to be the side of D_r intersecting the component V^+ (V^-) of $V - D_r$ containing the handles a_1 and a_2 (the handles a_3 and a_4).

The next observation is important. Suppose l is a loop in $T - V$ and as we go around l we count 1 or -1 (± 2 in the case of a_3) each time we pass through a handle of V , the sign being determined by whether we went through the handle with or against the direction of the small arrow next to the handle, for instance if we pass from a_3^+ to a_3^- on l add -2 . Using the right-hand rule we see that the sum total of the number of times l goes through the handles of V gives the algebraic linking number of l with respect to l_2 , denoted by $\text{Lk}(l, l_2)$. (See [1, p. 81] for a definition of algebraic linking.) Obviously the number of times l passes through D_r

(± 1 each time, depending on which direction one went through D_r) gives the algebraic linking number of l with respect to l_1 .

We first show $L \in G''$ and then show L is 1-linked.

3. $L \in G''$. Let S_0 be the surface (compact, orientable) having one boundary component and of genus 2. To show $L \in G''$ we show that there is a map f of S_0 into $T - V$ such that $f|_{\text{Bd } S_0}$ is a homeomorphism onto $\text{Bd } D_l$ and the image of the homology generators of S_0 belong in G' . We do not build f but rather we construct the desired image S of S_0 in $T - V$. We start with a disk with four holes S' as illustrated in Figure 3 (in Figure 3, V appears as a dotted 1-dimensional object

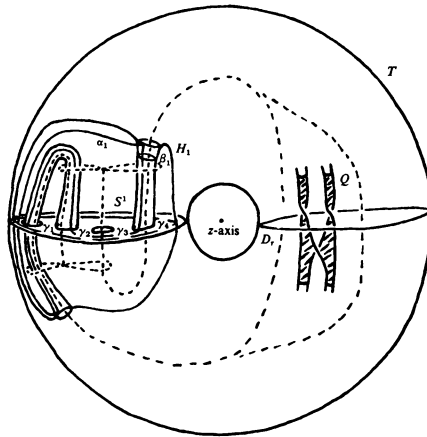


FIGURE 3

except at the right-hand side of the illustration). We add a handle H_1 (H_1 is an annulus) to S' by starting at γ_2 , following V up through its handle a_1 , down through the hole bounded by γ_1 , through handle a_3 , then, like a fountain, the annulus widens out, reverses direction in T and goes back through S' in a simple closed curve parallel to $\text{Bd } S' \cap \text{Bd } T$, then narrows down again around V , reverses direction in T , goes through a_2 and ends at γ_4 . (See Figure 3.) Notice the figure eight $\alpha_1\beta_1$ in $S' \cup H_1$ is such that $(\alpha_1\beta_1) \cap D_r = \emptyset$, hence $\text{Lk}(\alpha_1, l_1) = \text{Lk}(\beta_1, l_1) = 0$ and, again using the comment of §2, $\text{Lk}(\alpha_1, l_2) = 1 - 2 + 1 = 0$ and $\text{Lk}(\beta_1, l_2) = 0$. Hence $\alpha_1, \beta_1 \in G'$.

Since it would complicate Figure 3 too much to add the second handle H_2 , we just imagine it in Figure 3 as an annulus attaching γ_1 to γ_3 by going down through a_3 , up through S' in a simple closed curve parallel to $\text{Bd } S' \cap \text{Bd } T$, down through a_2 , γ_4 and up through a_4 . Again $S' \cup H_2$ contains a figure eight $\alpha_2\beta_2$ disjoint from $\alpha_1\beta_1$ such that $\text{Lk}(\alpha_2, l_1) = \text{Lk}(\beta_2, l_1) = \text{Lk}(\alpha_2, l_2) = \text{Lk}(\beta_2, l_2) = 0$ (where $\text{Lk}(\alpha_2, l_2) = -2 + 1 + 1 = 0$). Hence $\alpha_2, \beta_2 \in G'$. It can be checked that $S = S' \cup H_1 \cup H_2$ is the image of S_0 . It then follows that $\text{Bd } S$ is homotopic in $T - V$ to $(\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}) \cdot (\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1})$. Since $\alpha_1, \alpha_2, \beta_1, \beta_2 \in G'$, $\text{Bd } S \in G''$ and it follows that $l_1 \in G''$. Let

Q be the disk indicated in the right side of Figure 3, Q bounds a portion of l_2 . The four scc's making up $\text{Cl} [(l_2 - Q) \cup (\text{Bd } Q - l_2)]$ are homotopic to $\text{Bd } S$ in $T - l_2$ and it follows that $l_2 \in G''$.

It is interesting to note that the singularities of f on S_0 can be made to consist of two disjoint scc's parallel to $\text{Bd } S_0$ and two disjoint scc's in each of the two handles. Under f one meridian curve is sewed to the other and the remaining two meridian curves are sewed to the two curves parallel to $\text{Bd } S_0$.

4. L is 1-linked. The following nine lemmas combined with the assumption that l_1, l_2 bound disjoint orientable surfaces S_1, S'_2 , respectively, (i.e. L is not 1-linked) will be shown to lead to a contradiction. It should be noted that the linking properties developed in the following lemmas echo those of §3.

LEMMA 1. *If l_1, l_2 bound disjoint orientable surfaces S_1, S'_2 , respectively, then l_2 bounds an orientable surface S_2 such that $S_2 \subset \text{Int } T$, S_2 is in general position relative to D_r and at most one component of $l_2 - D_r$ is contained in a component of $S_2 - D_r$ intersecting both sides of D_r .*

Proof. By the existence of S_1 we may suppose $S'_2 \subset \text{Int } T$ and S'_2 is in general position relative to D_r . Let $l(i)$, $i = 1, 2, 3$ and 4 , be the component (open arc) of $l_2 - D_r$ going through the handle a_i of V and let $C(i)$ be the component of $S'_2 - D_r$ containing $l(i)$. Suppose $C(1)$ and $C(2)$ intersect both sides of D_r . If every component of $S'_2 - D_r$ which intersects D_r^+ also intersects D_r^- , then we could, by going around the components of $S'_2 - D_r$, find a loop l in S'_2 such that $\text{Lk}(l, \text{Bd } D_r) > 0$ (counting $+1$ each time we pass through D_r going from D_r^- to D_r^+). But from this it follows that $S_1 \cap S'_2 \neq \emptyset$, contradiction. Hence there is some component X of $S'_2 - D_r$ intersecting only D_r^+ . Then X separates $E^3 = \text{Int } T - D_r$ into exactly two components, one of which contains $C(1) \cup C(2)$; call this component $\text{Ext } X$, the other $\text{Int } X$. Replace X by the punctured disk (or disks) $\text{Cl}(\text{Int } X) \cap D_r$. (In this process we will also have to cut off all other parts of $S_1 \cup S'_2$ in $\text{Int } X$.) In any case by repeating this process a finite number of times, it follows that at least one of the resulting $C(1), C(2)$ intersects only D_r^+ .

By a similar reasoning relative to the components $C(3), C(4)$ we may suppose at least one of them intersects only D_r^- . By general position of S'_2 and D_r , either $C(1) = C(2)$ or $C(3) = C(4)$. Hence let $S_2 = S'_2$, and S_2 satisfies the conclusion of this lemma.

Let S be an orientable surface in $T - (D_r \cup V)$ such that $\text{Bd } S = \text{Bd } D_l$, $\text{Int } S \subset \text{Int } (T - (D_r \cup V))$ and S is in general position relative to $\bigcup_{i=1}^4 a_i$. If l is a loop in S , let $\text{Lk}(l, a_i)$ be the linking number of l with respect to the handle a_i only and $\text{Lk}(l, a_i \cup a_j)$ the linking number of l with respect to the handle a_i and a_j only.

We introduce now another set which will be useful in investigating the linking properties of S with respect to $\bigcup_{i=1}^4 a_i$. Let $T_A = T \cap xy\text{-plane}$, T_A is an annulus in T which intersects each a_i in two arcs (see Figure 2). Adjust S slightly in $T - V$ so that it is in general position relative to both T_A and $\bigcup_{i=1}^4 a_i$. Note that the

components of $(S \cap \text{Int } T_A) - \bigcup_{i=1}^4 a_i$ are scc's or open arcs whose closures are either arcs or scc's. Let $\Gamma = \{\gamma : \gamma \text{ is a component of } (S \cap T_A) - (\bigcup_{i=1}^4 a_i)\}$. We now alter S so that $S \cap (T_A \cup (\bigcup_{i=1}^4 a_i))$ is in a certain sense simpler. By adjusting S close to $\bigcup_{i=1}^4 a_i$ we may suppose that if l is a scc of $S \cap (\bigcup_{i=1}^4 a_i)$ that bounds a disk in $\bigcup_{i=1}^4 a_i$, then $l \cap T_A = \emptyset$ and if l does not bound a disk in $\bigcup_{i=1}^4 a_i$, then $l \cap T_A$ consists of exactly two points. Suppose $\gamma \in \Gamma$ and $\text{Cl } \gamma$ is an arc starting and ending on the same side of some a_i . We may then adjust S near T_A by pushing γ down through a_i , changing two (or more) scc's of $S \cap a_i$ to a scc l which bounds a disk on a_i and adjust S close to a_i so that $l \cap T_A = \emptyset$. Since this process reduces the number of scc's of $S \cap (\bigcup_{i=1}^4 a_i)$ which separate an a_i , it is finite. Now remove those scc's of $S \cap (\bigcup_{i=1}^4 a_i)$ which bound a disk in $\bigcup_{i=1}^4 a_i$ by cutting S off on $\bigcup_{i=1}^4 a_i$, let $S(r)$ be the component of the resulting surface which contains $\text{Bd } S$. Let $\Gamma(r) = \{\gamma : \gamma \text{ is a component of } (S(r) \cap T_A) - (\bigcup_{i=1}^4 a_i)\}$ and $\Delta(r) = \{\delta : \delta \text{ is a component of } S(r) - (\bigcup_{i=1}^4 a_i)\}$. Note that if $\text{Cl } \gamma$ is an arc intersecting both sides of an a_i , then we obtain a spiral in $S(r) \cap T_A$ going around a_i ; eventually it reverses itself and here we obtain an arc intersecting only one side of a_i , contradicting the form of $S(r)$. Therefore if $\gamma \in \Gamma(r)$ and $\text{Cl } \gamma$ is an arc, then its endpoints lie in different components of $\text{Bd } T \cup (\bigcup_{i=1}^4 a_i)$.

To say $\gamma \in \Gamma(r)$ ends on a_i^+ means γ approaches a_i from its positive side (denoted by $\gamma \cap a_i^+ \neq \emptyset$). Similarly we may define $\gamma \cap a_i^- \neq \emptyset$ and, for $\delta \in \Delta(r)$, $\delta \cap a_i^+ \neq \emptyset$ and $\delta \cap a_i^- \neq \emptyset$. We say two distinct elements γ_1, γ_2 of $\Gamma(r)$ abut on a_i if they approach a point of $T_A \cap (\bigcup_{i=1}^4 a_i)$ from opposite sides of a_i and $\delta_1, \delta_2 \in \Delta(r)$ abut on a_i if they approach a scc of $S \cap a_i$ from opposite sides of a_i . If γ ends on a_i , then by the form of $S(r)$, it follows that $S \cap a_i$ contains a scc l such that $l \cap T_A$ is two points b_1, b_2 where γ approaches b_1 . Let $\text{Op } \gamma(a_i^+)$ (or $\text{Op } \gamma(a_i^-)$) be the arc of $\Gamma(r)$ which approaches b_2 from the positive side of a_i , a_i^+ (or the negative side of a_i , a_i^-). It helps to draw in Figures 2, 4 and 5 the various arcs γ which arise in the next lemmas.

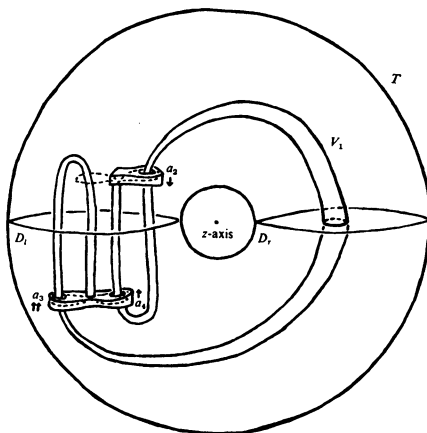


FIGURE 4

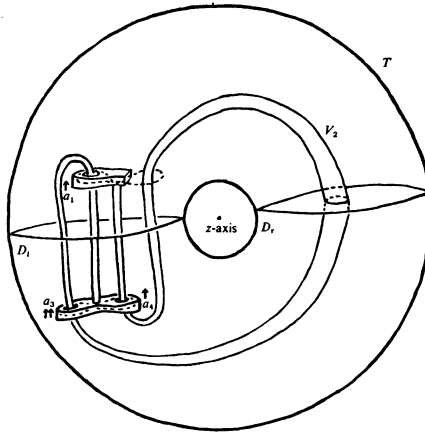


FIGURE 5

LEMMA 2. Suppose $\delta \in \Delta(r)$, $\delta \cap a_2^- \neq \emptyset$, $\text{Bd } S(r) \not\subset \delta$ and δ contains no $\gamma \in \Gamma(r)$ so that $\text{Cl } \gamma$ is a scc. Then $\delta \cap a_3^- \neq \emptyset$.

Proof. Suppose $\delta \cap a_3^- = \emptyset$. Since $\text{Bd } S(r) \not\subset \delta$ and δ contains no γ so that $\text{Cl } \gamma$ is a scc, there is an arc γ_1 of $\Gamma(r)$ in δ going from a_2^- to a_4^- . Let $\gamma_2 = \text{Op } \gamma_1(a_4^-)$, γ_2 must end on a_2^+ . Let $\gamma_3 = \text{Op } \gamma_2(a_2^+)$, $\gamma_3 \neq \gamma_1$ and γ_3 must end on a_4^- . Let $\gamma_5 = \text{Op } \gamma_4(a_4^-)$. Continuing in this manner, it follows that $\Gamma(r)$ must be an infinite set, contradiction. Hence γ_1 must end on a_3^- and $\delta \cap a_3^- \neq \emptyset$.

Also we have the following

LEMMA 2'. Suppose $\delta \in \Delta(r)$, $\delta \cap a_3^- \neq \emptyset$, $\text{Bd } S(r) \not\subset \delta$ and δ contains no $\gamma \in \Gamma(r)$ so that $\text{Cl } \gamma$ is a scc. Then $\delta \cap a_2^- \neq \emptyset$.

LEMMA 3. Suppose $\delta \in \Delta(r)$, $\delta \cap a_3^+ \neq \emptyset$, $\text{Bd } S(r) \not\subset \delta$ and δ contains no γ so that $\text{Cl } \gamma$ is a scc. Then $\delta \cap a_1^+ \neq \emptyset$.

Proof. Suppose $\delta \cap a_1^+ = \emptyset$. Since $\text{Bd } S(r) \not\subset \delta$ and δ contains no γ so that $\text{Cl } \gamma$ is a scc, there is an arc γ_1 of $\Gamma(r)$ in δ going from a_3^+ to a_2^- . Let $\gamma_2 = \text{Op } \gamma_1(a_2^-)$, by the proof of Lemma 2, γ_2 must end on a_3^- . Let $\gamma_3 = \text{Op } \gamma_2(a_3^-)$, $\gamma_3 \neq \gamma_1$, and γ_3 must end on a_2^- . Let $\gamma_4 = \text{Op } \gamma_3(a_2^-)$. Continuing in this manner, it follows that $\Gamma(r)$ must be an infinite set, contradiction. Hence γ_1 must end on a_1^+ and $\delta \cap a_1^+ \neq \emptyset$.

Also we have the following

LEMMA 3'. Suppose $\delta \in \Delta(r)$, $\delta \cap a_2^+ \neq \emptyset$, $\text{Bd } S(r) \not\subset \delta$ and δ contains no $\gamma \in \Gamma(r)$ so that $\text{Cl } \gamma$ is a scc. Then $\delta \cap a_4^- \neq \emptyset$.

The next lemma follows easily by examining the various arcs of $\Gamma(r)$ in δ .

LEMMA 4. Suppose $\delta \in \Delta(r)$, $\delta \cap a_1^- \neq \emptyset$, $\text{Bd } S(r) \not\subset \delta$ and δ contains no $\gamma \in \Gamma(r)$ so that $\text{Cl } \gamma$ is a scc. Then $\delta \cap a_2^+ \neq \emptyset$.

Also we have the following.

LEMMA 4'. Suppose $\delta \in \Delta(r)$, $\delta \cap a_4^+ \neq \emptyset$, $\text{Bd } S(r) \not\subset \delta$ and δ contains no $\gamma \in \Gamma(r)$ so that $\text{Cl } \gamma$ is a scc. Then $\delta \cap a_3^+ \neq \emptyset$.

LEMMA 5. There is a loop l in S so that either (1) $\text{Lk}(l, a_1 \cup a_2) \neq 0$ or (2) $\text{Lk}(l, a_3 \cup a_4) \neq 0$.

Proof. We first note that if there is a loop l' in $S(r)$ satisfying (1) or (2) of this lemma, then there is a loop l in S satisfying (1) or (2). For, while retaining property (1) or (2), we may push l' off the disks of $S(r)$ obtained by cutting S off on $\bigcup_{i=1}^4 a_i$. Pushing a $\text{Cl } \gamma$ down through some a_i does not alter the linking properties of l' with the various a_i . Hence reversing the steps needed to arrive at $S(r)$ from S , it follows that l' gives rise to an l in S satisfying condition (1) or (2).

If there is an element γ of $\Gamma(r)$ so that $\text{Cl } \gamma$ is a scc intersecting either a_1, a_2, a_3 or a_4 , then $l' = \text{Cl } \gamma$ satisfies condition (1) or (2). Suppose then no such scc exists.

Let δ_0 be the element of $\Delta(r)$ containing $\text{Bd } S(r)$ and suppose $\delta_0 \cap a_2^+ \neq \emptyset$. By Lemma 2, δ_0 abuts a δ_1 on a_2 so that $\delta_1 \cap a_3^- \neq \emptyset$ (assuming $\delta_0 \neq \delta_1$). By Lemma 3, δ_1 abuts a δ_2 on a_3 so that $\delta_2 \cap a_1^+ \neq \emptyset$ (again assuming $\delta_0 \neq \delta_2$). By Lemma 4, δ_2 abuts a δ_3 on a_1 so that $\delta_3 \cap a_2^+ \neq \emptyset$. Continuing this reasoning we obtain a sequence $\delta_0, \delta_1, \dots, \delta_i$ which eventually repeats an element of $\Delta(r)$. Let m be the first integer so that $n < m$, $\delta_n = \delta_m$ and $\delta_i \neq \delta_j$ if $i \neq j$ and $n < i, j < m$. If $m - n = 1$, then $\text{Cl}(\bigcup_{i=n}^m \delta_i)$ contains a loop l' so that $\text{Lk}(l', a_i) = \pm 1$ for some $i = 1, 2$ or 3 and $\text{Lk}(l', a_j) = 0$ for $j \neq i$. Hence l' satisfies either (1) or (2). If $m - n \geq 2$, then it follows that $\text{Cl}(\bigcup_{i=n}^m \delta_i)$ contains a loop l' so that $\text{Lk}(l', a_1 \cup a_2) \neq 0$.

If $\delta_0 \cap a_3^- \neq \emptyset$ or $\delta_0 \cap a_1^+ \neq \emptyset$, then we may start at either of these places and, repeating the same pattern as before, obtain a sequence $\delta_0, \dots, \delta_i$ so that $\text{Cl}(\bigcup_{i=n}^m \delta_i)$ contains a loop l' satisfying (1) or (2).

If $\delta_0 \cap a_2^- \neq \emptyset$, then making use of Lemmas 2', 3' and 4' it follows that $S(r)$ contains a loop l' satisfying (1) or (2). Since δ_0 must intersect one of a_2^+, a_3^-, a_1^+ , or a_2^- , Lemma 5 follows.

Let $V_i = V$ minus its i th handle, $i = 1, 2, 3$ and 4 , see Figures 4 and 5 for V_1 and V_2 , respectively. Let S_i be a surface in $T - (D_r \cup V_i)$ such that $\text{Bd } S_i = \text{Bd } D_i$, $\text{Int } S_i \subset \text{Int}(T - (D_r \cup V_i))$ and S_i is in general position relative to $(\bigcup_{j=1}^4 a_j) - a_i$. As before, from S_i we may obtain a surface $S_i(r)$ so that for each arc γ of $\Gamma_i(r) = \{\gamma : \gamma \text{ is a component of } (S_i \cap T_A) - ((\bigcup_{j=1}^4 a_j) - a_i)\}$ such that $\text{Cl } \gamma$ is not a scc we have that its endpoints lie in different components of $\text{Bd } T \cup ((\bigcup_{j=1}^4 a_j) - a_i)$.

LEMMA 6. There is a loop l in S_1 so that either (1) $\text{Lk}(l, a_2) \neq 0$ or (2)

$$\text{Lk}(l, a_3 \cup a_4) \neq 0.$$

Proof. As in Lemma 5, we note that if $S_1(r)$ contains a loop satisfying (1) or (2) then so does S_1 . If there is an element γ of $\Gamma_1(r)$ so that $\text{Cl } \gamma$ is a scc, then $l' = \text{Cl } \gamma$ satisfies (1) or (2) of this lemma. Suppose then no such scc exists. Note that $S_1(r) \cap a_3 = \emptyset$ and $S_1(r) \cap a_4 \neq \emptyset$. There is an element γ_1 of $\Gamma_1(r)$ going from a_4^+ to a_2^+ . Let $\gamma_2 = \text{Op } \gamma_1(a_2^+)$. If γ_2 ends on $\text{Bd } T$, then the arc γ_2 abuts on a_2 would

start and end on a_2^- , contradiction. Hence γ_2 must end on a_4^- . Let $\gamma_3 = \text{Op } \gamma_2(a_4^-)$. Continuing in this manner, it follows that $\Gamma_1(r)$ must be infinite, contradiction.

The proof of the next lemma is (geometrically) analogous to the proof of Lemma 6.

LEMMA 7. *There is a loop l in S_4 so that either (1) $\text{Lk}(l, a_3) \neq 0$ or (2)*

$$\text{Lk}(l, a_1 \cup a_2) \neq 0.$$

LEMMA 8. *There is a loop l in S_2 so that either (1) $\text{Lk}(l, a_1) \neq 0$ or (2)*

$$\text{Lk}(l, a_3 \cup a_4) \neq 0.$$

Proof. Again note that if $S_2(r)$ contains a loop satisfying (1) or (2), then so does S_2 , hence assume no $\text{Cl } \gamma$ is a scc. Since $S_2 \cap a_1 \neq \emptyset$, there are arcs γ', γ'' in $\Gamma_2(r)$ so that both γ', γ'' start on a_1^- , γ' ends on a_3^+ , γ'' ends on a_4^+ and γ', γ'' end on the innermost scc of $S_2(r) \cap a_1$ (see Figure 5). Let γ'_0, γ''_0 be the arcs γ', γ'' about on a_3, a_4 , respectively. Since not both γ'_0, γ''_0 can end on $\text{Bd } T$, it follows that $\gamma'_0 = \gamma''_0$ and the scc $l = \text{Cl}(\gamma' \cup \gamma'' \cup \gamma'_0)$ plus an arc in $S_2(r) \cap a_1$ satisfies (2) of this lemma.

We also have the following

LEMMA 9. *There is a loop l in S_3 so that either (1) $\text{Lk}(l, a_4) \neq 0$ or (2)*

$$\text{Lk}(l, a_1 \cup a_2) \neq 0.$$

THEOREM. *L is 1-linked.*

Proof. Suppose l_1, l_2 bound disjoint orientable surfaces S_1, S'_2 , respectively. Let $l(i), C(i), i = 1, 2, 3$, and 4 be as given in the proof of Lemma 1. By Lemma 1, l_2 bounds an orientable surface S_2 such that $S_2 \subset \text{Int } T$, S_2 is in general position relative to D_r and at most one component $C(i)$ of $S_2 - D_r$ intersects both sides of D_r . Suppose this $i = 1$.

Let U be a closed regular neighborhood of $C(3) \cup C(4) \cup D_r$ chosen so $\text{Bd } U$ contains a surface S such that $\text{Int } S \subset \text{Int}(T - (D_r \cup l_2))$ and $\text{Bd } S = \text{Bd } D_r$. Further, U may be chosen so that $S \cap C(i) = \emptyset$ for $i = 2, 3$ and 4, since these $C(i)$'s intersect just one side of D_r . Since $S \cap l_2 = \emptyset$, we may push S off V (we may need to use a cut and paste argument on $S \cup (\bigcup_{i=2}^4 C(i))$ to get S off the handle of V intersecting a_3 since l_2 goes through this handle twice).

There are disjoint arcs x_1, x_2 in $\text{Cl}(C(3) \cup C(4)) \cap D_r$ so that either (1) $l' = l(3) \cup l(4) \cup x_1 \cup x_2$ is a scc or (2) $l(3) \cup x_1$ and $l(4) \cup x_2$ are scc's. In Case (2) let D be a disk in $\text{Int } D_r$ so that $x_1 \cup x_2 \subset \text{Bd } D$ and the arcs x'_1, x'_2 forming $\text{Bd } D - \text{Int}(x_1 \cup x_2)$ form a scc $l'' = l(3) \cup l(4) \cup x'_1 \cup x'_2$ which admits an orientation compatible with the orientation on $l(3)$ and $l(4)$ induced by the orientation of l_2 . Let $d_n, n = 1, 2, \dots, m$ be the disks in D_r bounded by the scc's of $\text{Cl}(C(3) \cup C(4)) \cap D_r$. Let $C' = \text{Cl}(C(3) \cup C(4)) \cup (\bigcup_{n=1}^m d_n)$ and $C'' = C' \cup D$. It then follows that $l' \sim 0$ in C' and $l'' \sim 0$ in C'' (using integer coefficients). Since $S \cap C' = S \cap C'' = \emptyset$, by Alexander Duality for each loop l in S , $\text{Lk}(l, a_3 \cup a_4) = 0$.

If both $C(1)$, $C(2)$ intersect just one side of D_r , then by a similar argument as used above, it follows that for each loop l in S , $\text{Lk}(l, a_1 \cup a_2) = 0$. But this is impossible by Lemma 5. If $C(1)$ intersects both sides of D_r , then, since $C(2)$ intersects only one side of D_r , $C(1) \cap C(2) = \emptyset$ and it follows that for each loop l in S , $\text{Lk}(l, a_2) = 0$, but this contradicts Lemma 6. The cases $i=2, 3$ and 4 are similar. Hence the surfaces S_1, S'_2 could not have existed and L is 1-linked.

5. Concluding remarks. Since our example $L = l_1 \cup l_2$ is 1-linked it follows that if l_1 bounds the orientable surface S_1 in $S^3 - l_2$, then S_1 contains a loop l such that $l \sim 0$ in $H_1(S^3 - l_2)$. In particular we have the following

THEOREM. *If $K = k_1 \cup k_2$ is a link of two components, then K is a boundary link if and only if k_1 bounds an orientable surface S_1 in $S^3 - k_2$ such that the inclusion of $H_1(S_1)$ in $H_1(S^3 - k_2)$ is trivial.*

Proof. The only if part follows immediately from the definition of boundary link. If the surface S_1 exists then k_2 bounds an orientable surface S_2 in $S^3 - k_1$. Put S_2 in general position with respect to the homology generators H (figure eights) of S_1 . Since the inclusion of $H_1(S_1)$ in $H_1(S^3 - k_2)$ is trivial, we may add handles to S_2 minus a small regular neighborhood of H to form an orientable surface S'_2 such that $\text{Bd } S'_2 = k_2$ and $S'_2 \subset S^3 - (k_1 \cup H)$. Since $S_1 - H$ is a disk minus a finite number of points we may put S'_2 in general position relative to $S_1 - H$ and cut S'_2 off on S_1 . We then obtain an orientable surface S''_2 such that $\text{Bd } S''_2 = k_2$ and $S_1 \cap S''_2 = \emptyset$. Hence K is a boundary link.

This theorem and our example L motivate the following

QUESTION. Does there exist a link $K = k_1 \cup \dots \cup k_n$, $2 < n$, such that each k_i bounds an orientable surface S_i in $S^3 - (K - k_i)$, and the inclusion of $H_1(S_i)$ in $H_1(S^3 - (K - k_i))$ is trivial but K is 1-linked?

Such a link K of the question would also be an example of a 1-linked link whose longitudes all lie in the second commutator subgroup.

REFERENCES

1. P. S. Aleksandrov, *Combinatorial topology*, Vol. 3, Graylock, New York, 1960.
2. J. Andrews and M. Curtis, *Knotted 2-spheres in the 4-sphere*, Ann. of Math. **70** (1959), 565-571.
3. S. Eilenberg, *Multicoherence*. I, II, Fund. Math. **27** (1936).
4. R. H. Fox, "A quick trip through knot theory" in *Topology of 3-manifolds*, Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 120-167.
5. N. Smythe, "Boundary links" in *Topology seminar, Wisconsin*, 1965, Annals of Mathematics Studies, no. 60, Princeton Univ. Press, Princeton, N. J., 1966, pp. 69-72.

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